Math 010 Exam 2
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. Let the transformation T be defined by $T(x_1, x_2, x_3) = (2x_1 - x_2 + 3x_3,\ x_1 + 4x_2 - x_3)$.
a. (3 pts) Find the standard matrix $[T]$.

$$[T] = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \end{bmatrix}$$

b. (1 pt) State the domain and codomain of T.

Domain: R^3

Codomain: R^2

c. (2 pts) Use the matrix you found in part (a) to compute $T(1, -1, 2)$.

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1+6 \\ 1-4-2 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

2. (6 pts) Use the linearity conditions to determine whether $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (2x_1 - x_2,\ x_1 + 3x_2,\ x_1^2)$ is a linear transformation. Be sure to justify your answer.

ii) Is $T(k\vec{x}) = kT(\vec{x})$?

$T(kx_1, kx_2) = (2kx_1 - kx_2,\ kx_1 + 3kx_2,\ k^2x_1^2)$

But $kT(x_1, x_2) = (2kx_1 - kx_2,\ kx_1 + 3kx_2,\ kx_1^2)$

$\neq T(k\vec{x})$ so no.

3. a. (6 pts) Find the standard matrix for the stated composition in R^2.
The orthogonal projection onto the y-axis, followed by a counterclockwise rotation by 90°.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_{90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R_{90°} A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

b. (2 pts) Compute the image of $(3, -4)$ under this transformation.

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

4. (8 pts) Find all values of λ for which $\det(A) = 0$, where

$$A = \begin{bmatrix} \lambda - 5 & 2 & 0 \\ 0 & \lambda - 3 & 0 \\ 1 & 2 & \lambda - 1 \end{bmatrix}$$

$$(\lambda - 1) \begin{vmatrix} \lambda - 5 & 2 \\ 0 & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 8\lambda - 15) = 0$$

$$(\lambda - 1)(\lambda - 5)(\lambda - 3) = 0$$

$$\lambda = 1, 3, 5$$

5. (10 pts) Find the polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through the three points $(1,0)$, $(2,3)$, and $(3,10)$ by row reducing an appropriate augmented matrix. (*Note: You don't need to take it all the way to RREF or even row echelon form. Row reduce until you can easily obtain the answer.*)

$(1,0)$: $a_0 + a_1 + a_2 = 0$

$(2,3)$: $a_0 + 2a_1 + 4a_2 = 3$

$(3,10)$: $a_0 + 3a_1 + 9a_2 = 10$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 10 \end{array}\right]$$

$R_2 \Rightarrow R_2 - R_1$

$$\begin{array}{cccc} 1 & 2 & 4 & 3 \\ -1 & -1 & -1 & 0 \\ \hline 0 & 1 & 3 & 3 \end{array}$$

$R_3 \Rightarrow R_3 - R_1$

$$\begin{array}{cccc} 1 & 3 & 9 & 10 \\ -1 & -1 & -1 & 0 \\ \hline 0 & 2 & 8 & 10 \end{array}$$

$\Rightarrow$ use 0 1 4 5

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

$R_3 \Rightarrow R_3 - R_1$

$$\begin{array}{cccc} 0 & 1 & 4 & 5 \\ 0 & -1 & -3 & -3 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow$$

$a_0 - 3 + 2 = 0 \Rightarrow a_0 = 1$

$a_1 + 6 = 3 \Rightarrow a_1 = -3$

$a_2 = 2$

$$\boxed{y = 1 - 3x + 2x^2}$$

6. (2 pts each) Suppose A is a 3×3 matrix with $\det(A) = 4$. Compute each of the following.

a. $\det(A^{-1})$

$$\frac{1}{\det(A)} = \frac{1}{4}$$

b. $\det(3A)$

$$3^3 \det(A) = 108$$

c. $\det(-A)$

$$(-1)^3 \det(A) = -4$$

d. $\det(A^T A)$

$\det(A^T) = \det(A)$ so $4 \cdot 4 = 16$

7. (6 pts) Find a and b such that $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$, where $\mathbf{u} = (1, 3, -2)$, $\mathbf{v} = (2, 1, 1)$, and $\mathbf{w} = (5, 5, -1)$ or show that no such scalars exist.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 1 & 5 \\ -2 & 1 & -1 \end{array}\right]$$

$R_2 \Rightarrow R_2 - 3R_1$

$$\begin{array}{ccc} 3 & 1 & 5 \\ -3 & -6 & -10 \\ \hline 0 & -5 & -5 \end{array}$$

$\Rightarrow 0 \ 1 \ 1$

$R_3 \Rightarrow R_3 + 2R_1$

$$\begin{array}{ccc} -2 & 1 & -1 \\ 2 & 4 & 10 \\ \hline 0 & 5 & 9 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 5 & 9 \end{array}\right]$$

$R_3 \Rightarrow R_3 - 5R_2$

$$\begin{array}{ccc} 0 & 5 & 9 \\ 0 & -5 & -5 \\ \hline 0 & 0 & 4 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

$0 = 4$ is a contradiction.

Such a & b do not exist.

8. (2 pts each) Let $\mathbf{u} = (1, 2, 2)$ and $\mathbf{v} = (2, 1, -1)$. Compute each of the following:

a. $\|3\mathbf{u} - 2\mathbf{v}\|$

$$3\vec{u} - 2\vec{v} = (3, 6, 6) - (4, 2, -2) = (-1, 4, 8)$$

$$\|3\vec{u} - 2\vec{v}\| = \sqrt{1 + 16 + 64} = 9$$

b. $\mathbf{u} \cdot \mathbf{v}$

$$= 1(2) + 2(1) + 2(-1) = 2$$

c. Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? Please justify your answer.

No because $\vec{u} \cdot \vec{v} \neq 0$.

d. Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{2}{3\sqrt{6}}$$

9. (4 pts) Let $\mathbf{u} = (4, 1, 6)$ and $\mathbf{a} = (1, 2, 2)$. Find the following:
a. The vector projection of $\mathbf{u}$ onto $\mathbf{a}$, $\text{proj}_{\mathbf{a}}\mathbf{u}$.

$$\text{proj}_{\vec{a}}\,\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,a = \frac{4+2+12}{9}\,\vec{a}$$

$$= 2\vec{a} = (2,4,4) = \vec{w}_1$$

b. (3 pts) The component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ and verify that it is orthogonal to $\mathbf{a}$.

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = (4,1,6) - (2,4,4)$$

$$= (2,-3,2)$$

$$\vec{w}_2 \cdot \vec{a} = (2,-3,2)\cdot(2,4,4)$$

$$= 4 - 12 + 8 = 0 \checkmark$$

$$\vec{w}_2 \perp \vec{a}$$

c. (3 pts) Confirm that the results from (a) and (b) add up to $\mathbf{u}$.

$$(2,-3,2) + (2,4,4) = (4,1,6) = \vec{u} \checkmark$$

10. (8 pts) Prove following property of the dot product: $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$

Let $\vec{u} = (u_1, u_2, \dots, u_n)$, $\vec{v} = (v_1, v_2, \dots, v_n)$, $k \in \mathbb{R}$.

$k(\vec{u}\cdot\vec{v})$

$= k(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$ by def. of dot product

$= ku_1v_1 + ku_2v_2 + \cdots + ku_nv_n$ by the distributive property for real #s

$= (ku_1)v_1 + (ku_2)v_2 + \cdots + (ku_n)v_n$ by the associative property for multiplication of real #s

$= (k\vec{u})\cdot\vec{v}$ ✓ by def. of dot product.

Previously established. Included here for completeness.

11. (10 pts) Prove that linear transformations $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$ commute if and only if the matrices A and B commute.

$(T_A \circ T_B)(\vec{x}) = T_A(T_B(\vec{x})) = T_A(B\vec{x}) = A(B\vec{x})$

$= (AB)\vec{x}$. Thus $(T_A \circ T_B)(\vec{x}) = T_{AB}(\vec{x})$.

T_A & T_B commute $\Leftrightarrow (T_A \circ T_B)(\vec{x}) = (T_B \circ T_A)(\vec{x})$

$\Leftrightarrow T_{AB}(\vec{x}) = T_{BA}(\vec{x})$

$\Leftrightarrow AB = BA$

$\Leftrightarrow A$ & B commute. ✓